Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Maximum Marks: 60, Duration: Three hours
Date: 29.04.2019, 2:30 PM - 5: 30 PM

- Answer as many questions as you can.
- Maximum marks is 60.
- You may consult any book or course notes during the exam.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Let G be a topological group with multiplication $m : G \times G \to G$. Let ω denote the composite

$$G \vee G \subset G \times G \xrightarrow{m} G.$$

Calculate the effect of ω on homology groups using the isomorphism $\tilde{H}_*(G \vee G) \cong \tilde{H}_*(G) \oplus \tilde{H}_*(G)$. 6

2. a) Consider $TS^2 = \{(x, v) | x \in S^2, v \in \mathbb{R}^3, \langle v, x\rangle = 0\} \subset S^2 \times \mathbb{R}^3$. Show that TS^2 deformation retracts to $A = \{(x, 0) | x \in S^2\} \subset TS^2$, which is homeomorphic to S^2. 3
b) Let $X = \{(x, v) \in TS^2 || v| = 1\}$. Show that $TS^2 - A$ deformation retracts to X. 3
c) Consider $\pi_1 : TS^2 \to S^2$ the projection onto the first factor. Show that any $s : S^2 \to TS^2$ such that $\pi_1 \circ s = id$, satisfies $Im(s) \cap A \neq \phi$. 4

3. Let $\Delta : S^n \to S^n \times S^n$ be the diagonal map given by $\Delta(x) = (x, x)$. Calculate $H_*(\text{Cone}(\Delta))$, the homology of the mapping cone of Δ. 8

4. Let $q : S^n \to \mathbb{R}P^n$ be the usual quotient map.
a) Compute $q_* : H_n(S^n) \to H_n(\mathbb{R}P^n)$. 5
b) Compute $q_* : H_n(S^n; \mathbb{Z}/2) \to H_n(\mathbb{R}P^n; \mathbb{Z}/2)$. 5

5. Let X be a CW complex, and Σ_2 be the surface of genus 2. Prove that $H_k(X \times \Sigma_2) \cong H_k(X) \oplus H_{k-1}(X)^4 \oplus H_{k-2}(X)$. 7

6. Let X be a path connected space such that $\pi_1(X)$ is a finite group. Prove that for any map $X \to S^1 \times S^1$, the induced map $f_* : \tilde{H}_*(X) \to \tilde{H}_*(S^1 \times S^1)$ is 0. 7

7. a) For any map $f : S^{2n} \to S^{2n}$, prove that $\exists\, x \in S^{2n}$ with $f(x) = x$, that is, f has a fixed point. 2
b) Prove that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. 5
c) Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points. 3

8. Let $L_3(k)$ be the Lens space S^3/C_k.
a) Using the usual CW complex structure on $L_3(k)$ and $\mathbb{R}P^2$, and the induced CW complex structure on $L_3(k) \times \mathbb{R}P^2$, write down the cellular chain complex of $L_3(k) \times \mathbb{R}P^2$. 6
(b) Compute $H_*(L_3(k) \times \mathbb{R}P^2)$. 4
b) Prove that any CW complex homotopic to $L_3(k) \times \mathbb{R}P^2$ must have dimension ≥ 5. 3