$$\Rightarrow \quad k \leq n - \log_q \sum_{i=0}^{e} \binom{n}{i} (q-1)^i$$

$\Rightarrow \forall (n,k,d)_q$ code C,

$$R \lesssim 1 - \frac{\log_q \left(\sum_{i=0}^{e} \binom{n}{i} (q-1)^i \right)}{n}.$$

$$= 1 - \frac{Vol_q(\lfloor \frac{d-1}{2} \rfloor, n)}{n}.$$

Defn (Perfect Code): Any code that achieves Hamming bound is called the perfect code.

- Optimal trade-off b/w redundancy & error-correction capability for code-families:

Code family: $C = \{C_i\}_{i \in \mathbb{N}}$ where $C_i = (n_i, k_i, d_i)_q$

& relative distance $\delta(C) = \lim_{i \to \infty} \frac{d_i}{n_i}$, when exists

rate of C, $R(C) = \lim_{i \to \infty} \frac{k_i}{n_i}$, when exists.

Motivating Question 2: What is the optimal trade-off b/w $R(C)$ & $\delta(C)$ that can be achieved by some code family C?

We try to answer this question. To do that first we need introduce a fundamental function, called entropy function, which plays a crucial role in studying trade-off b/w $R(C)$ & $\delta(C)$.

=: Trade-off b/w Rate & Distance-II :=

Proposition 1: Let $q \geq 2$ be an integer & $p \in [0, 1-\frac{1}{q})$. Then,

i) $Vol_q(pn, n) \leq q^{H_q(p)n}$ &

ii. for large enough n, $Vol_q(pn, n) \geq q^{H_q(p)n - o(n)}$.

Proof:

$$1 = (p + (1-p))^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} p^i \cdot (1-p)^i$$

$$= \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^i + \sum_{i=pn+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$\geq \sum_{i=0}^{pn} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{pn} \binom{n}{i} (q-1)^i \left(\frac{p}{q-1}\right)^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{pn} \binom{n}{i} \cdot (q-1)^i \cdot (1-p)^n \cdot \left[\frac{p}{(q-1)(1-p)}\right]^i$$

$$\boxed{\begin{aligned} &\text{Since } p \in (0, 1-\tfrac{1}{q}], \quad p \leq 1-\tfrac{1}{q} \Rightarrow \frac{p}{(q-1)} \leq \frac{1}{q} \\ &\qquad\qquad\qquad\qquad\quad \Downarrow \\ &\qquad\qquad\qquad\quad p-1 \leq -\tfrac{1}{q} \qquad \Rightarrow \frac{p}{(q-1)(1-p)} \leq 1 \\ &\qquad\qquad\qquad\quad (1-p) \geq \tfrac{1}{q} \end{aligned}}$$

This implies that

$$1 \geq \sum_{i=0}^{pn} \binom{n}{i} \cdot (q-1)^i (1-p)^n \cdot \left[\frac{p}{(1-p)(q-1)}\right]^{pn}$$

$$= \left[\sum_{i=0}^{pn} \binom{n}{i} (q-1)^i\right] \frac{p^{pn} (1-p)^{(1-p)pn}}{(q-1)^{pn}}$$

Observe, $H_q(p) = p \cdot \log_q q-1 - p\log_q p - (1-p)\log_q(1-p)$

$$\Rightarrow q^{H_q(p)} = (q-1)^p \cdot p^{-p} \cdot (1-p)^{-(1-p)}$$

$$\Rightarrow q^{-H_q(p)n} = \left(\frac{p}{q-1}\right)^{pn} \cdot (1-p)^{pn}$$

$$\Rightarrow 1 \geq \mathrm{Vol}_q(pn, n) \cdot q^{-H_q(p)n}$$

$$\Rightarrow \mathrm{Vol}_q(pn, n) \leq q^{H_q(p) \cdot n}$$

(ii) Stirling Approximation: $\forall\ n \in \mathbb{N}$,

$$\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot e^{\lambda_1(n)} < n! < \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot e^{\lambda_2(n)},$$

where. $\lambda_1(n) = \frac{1}{12n+1}$ & $\lambda_2(n) = \frac{1}{12n}$.

By Stirling's approximation,

$$\binom{n}{pn} = \frac{n!}{(pn)!\,((1-p)n)!}$$

$$> \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{pn}{e}\right)^{pn} \cdot \left((1-p)\cdot\frac{n}{e}\right)^{(1-p)n}} \cdot \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot e^{\lambda_1(n) - \lambda_2(pn) - \lambda_2((1-p)n)}$$

$$= \frac{1}{p^{pn} \cdot (1-p)^{(1-p)n}} \cdot \ell(n), \text{ where } \ell(n) = \frac{e^{\lambda_1(n) - \lambda_2(pn) - \lambda_2(1-p)}}{\sqrt{2\pi p(1-p)n}}$$

$$\Rightarrow \mathrm{Vol}_q(pn, n) \geqslant \binom{n}{pn}(q-1)^{pn}$$

$$> \left[\frac{(q-1)^p}{p^p(1-p)^{(1-p)}}\right]^n \cdot \ell(n)$$

$$\geqslant q^{H_q(p)\cdot n - o(n)}$$