PRACTICE PROBLEMS I: FOURIER ANALYSIS 2025

(1) Find the volume of an ellipsoid in $\mathbb{R}^n$ with the lengths of its axes $a_1, a_2, \ldots, a_n$.

(2) Verify if this is true: $L^{p_0,\infty}(\mathbb{R}^n) \cap L^{p_1,\infty}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $1 \le p_0 < p < p_1 \le \infty$. (Note that $L^{\infty,\infty} = L^\infty$.)

(3) Suppose $f \in L^1(\mathbb{R})$ satisfies $f(x) = 0$ for $x < 0$. Find the domain in the complex plane where $\widehat{f}$ can be extended as a holomorphic function.

(4) Suppose for a function $f \in L^1(\mathbb{R})$ its Fourier transform $\widehat{f}$ satisfies $|\widehat{f}(x)| \le G_t(x)$ for a fixed $t > 0$, for all $x \in \mathbb{R}$. Show that f is infinitely differentiable. Here $G_t(x) = e^{-\pi t|x|^2}$

(5) Suppose that for two functions $f_1, f_2 \in S(\mathbb{R})$ and a polynomial P, $P(d/dx)f_1 = P(d/dx)f_2$. Then show that $f_1 = f_2$. If instead $f_1, f_2 \in L^p(\mathbb{R})$ for $p \in [1,2]$ and they satisfy the same condition (where derivatives are in the sense of distribution), what is your conclusion regarding equality of f_1 and f_2?

(6) Find Fourier transform of Af where $A \in \mathrm{GL}(n,\mathbb{R})$ and $f \in S(\mathbb{R}^n)$.

(7) Use Hausdorff-Young theorem to define Fourier transform of a function $f \in L^p(\mathbb{R})$ where $1 < p < 2$. Recall that Fourier transform of this f can also be defined by writing $f = f_1 + f_2$ where $f_1 \in L^1$ and $f_2 \in L^2$. Show that these two Fourier transforms are same as measurable functions.

(8) Let T be a tempered distribution and $\psi \in S(\mathbb{R})$. Define the following tempered distributions: dilation T_δ, product ψT and $T * \psi$. Show that $\widehat{T * \psi} = \widehat{T}\widehat{\psi}$.

(9) Assume that $f \in L^1(\mathbb{R}^n)$ is such that $\widehat{f} \in L^1(\mathbb{R}^n)$. Check if the proof of inversion formula of Fourier transform works for this f. (In that formula we used $f \in S(\mathbb{R}^n)$.)

(10) Let A, B be two measurable subsets of $\mathbb{R}^n$ of finite measures. Suppose $|A+B|^\gamma \ge C|A|^\alpha\,|B|^\beta$ for some numbers $0 < \alpha, \beta, \gamma < 1$. Show that $\alpha + \beta = \gamma$. Here by $|A|$ we mean Lebesgue measure of A.

(11) Recall that Fourier transform of a radial function is radial. Suppose $f \in L^1(\mathbb{R}^2)$ satisfies $f(k_\theta x) = e^{im\theta} f(x)$ for all $x \in \mathbb{R}^2$ where $m \in \mathbb{Z}$ is fixed and

$$k_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Find $\widehat{f}(k_\theta \xi)$ in terms of $\widehat{f}(\xi)$.