PRACTICE PROBLEMS 2: FOURIER ANALYSIS 2025

0.1. **Problems on circle.**

(1) Fix a $p \geq 1$. Show that there is no $f \in L^q(\mathbb{T})$ for any $q \geq 1$ such that $f * g = g$ for every $g \in L^p(\mathbb{T})$.

(2) Consider the following function on $\mathbb{T}$:

$$f(\theta) = i(\pi - \theta) \text{ for } \theta \in [0, \pi], f(\theta) = -f(-\theta) \text{ for } \theta \in (-\pi, 0), f(-\pi) = f(\pi).$$

Find the Fourier series of f.

(3) Suppose that for a function f on $\mathbb{T}$, $\widehat{f}(n) = 1/n^\alpha$ for some $0 < \alpha < 1$ if $n > 0$ and $\widehat{f}(n) = 0$ for $n \leq 0$. Find $p \geq 1$ such that $f \notin Ł^p(\mathbb{T})$.

(4) For $1 < p < \infty$, suppose for an L^p function f on $\mathbb{T}$, $S_N f(x) \to g(x)$ point-wise a.e. for some measurable function g on $\mathbb{T}$. Then $f = g$. What happens if $p = 1$?

(5) Suppose for a function $f \in L^1(\mathbb{T})$, $\widehat{f}(n) = 1/n$ for $n > 0$ and $\widehat{f}(n) = 0$ for $n \leq 0$. Show that f is not continuous. Follow these steps:

Find its Abel sum $A_r(f)(\theta)$ at $\theta = 0$, $0 < r < 1$. Find $\lim_{r \to 1} A_r f(0)$ and give your argument.

(6) In this exercise by an ideal of the algebra $L^1(\mathbb{T})$, we mean a closed convolution-ideal. Take two $n, m \in \mathbb{Z}$. Show that

$$I_{m,n} = \{f \in L^1(\mathbb{T}) \mid \widehat{f}(n) = 0 \ \& \ \widehat{f}(m) = 0\}$$

is an ideal of $L^1(\mathbb{T})$, which is not a maximal ideal.

(7) Show that a closed convolution-ideal I of $L^1(\mathbb{T})$ is invariant under translations.

(8) Take $f_1, f_2 \in C(\mathbb{T})$. Suppose that Fourier series of f_2 is absolutely convergent and the inversion holds for f_2, i.e.

$$f_2(\theta) = \sum_{n=-\infty}^{\infty} \widehat{f_2}(n) e^{in\theta}, \forall \theta \in \mathbb{T}.$$

Let $g(\theta) = f_1(\theta) f_2(\theta)$. Show that

$$\widehat{g}(m) = \sum_{n=-\infty}^{\infty} \widehat{f_1}(m-n) \widehat{f_2}(n).$$

(Explain that the series in rhs makes sense and note that rhs is convolution of two sequences.)

(9) For a function $f \in L^1(\mathbb{T})$ let

$$I(f) = \{f * g \mid g \in L^1(\mathbb{T})\} \text{ and } V(f) = \text{Span}\{\tau_\alpha f \mid \alpha \in [0, 2\pi]\},$$

where $\tau_\alpha f(\theta) = f(\theta - \alpha)$. Show that closure of $V(f)$ is equal to the closure of $I(f)$.

0.2. **Additional problems on $\mathbb{R}^n$.**

(1) Find the range of α for which $1/|x|^\alpha$ is a tempered distribution for $x \in \mathbb{R}^n$.

(2) Fix a $\xi_0 \in \mathbb{R}$. What is the Fourier transform of $x \mapsto e^{2\pi i \xi_0 x}$?

(3) If a linear operator T is strong-type $p - q$, then show that T^* is strong-type $q' - p'$.

(4) Define a multiplier operator T on Schwartz space functions and show that it is translation invariant.

(5) Verify if the translation by a fixed element $x \in \mathbb{R}^n$ is a multiplier operator.

(6) Fix a function f and define a linear operator T by $T = T_f : g \mapsto g * f$. Find T^*.