**Homework 1** *Martingale Theory, S. S. Mukherjee, Fall 2025*

Topics: Stopping times, LLN Due on Oct 31, 2025

Name of student:
Roll number:

### 1. Distribution of the exit time of symmetric intervals [4]

Let $\xi_i, i \geq 1$, be i.i.d. Rademacher random variables (i.e. random variables taking values $\pm 1$ w.p. $1/2$ each). Consider the simple symmetric random walk $(S_n)_{n \geq 0}$ defined recursively as follows: $S_0 = 0$ and $S_n = S_{n-1} + \xi_n$ for $n \geq 1$. For $a \in \mathbb{N}$, define $\tau_a := \inf\{n \geq 1 \mid S_n \notin [-a, a]\}$ to be the first exit time of the interval $[-a, a]$. Compute the probability generating function of $\tau_a$.

### 2. A law of large numbers [6]

Let $(M_n)_{n \geq 0}$ be a square-integrable martingale and for $n \geq 1$, let $\Delta M_n := M_n - M_{n-1}$. Suppose that for some sequence $b_n \uparrow \infty$, we have $\sum_{k \geq 1} b_k^{-2} \mathbb{E}(\Delta M_k)^2 < \infty$. Prove that $b_n^{-1} M_n \stackrel{\text{a.s.}}{\to} 0$.