Assignment 1

Probability Theory (M. Math.)

The assignment is due on **19/08/25**. In case of queries/comments, email me.

Recall that $(X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E}))_i$ are i.i.d. with law μ if X_i are independent and $(X_i)_* \mathbb{P} = \mu$.

1. Suppose X is an action with outcomes measured in $(E, \mathcal{E}, \mu)$. Show that the action of repeating X k-times, each time independently of the others, is modeled by the coordinate functions in $(E^k, \otimes_{i=1}^k \mathcal{E}, \otimes_{i=1}^k \mu)$, that is, the coordinate functions are i.i.d. with law μ. For any finite subset $S \subset \mathbb{N}$, define $\mu_S := \bigotimes_S \mu$. Show that for $S' \subset S$, and the projection $\pi_{S \to S'} : E^S \to E^{S'}$, $(\pi_{S \to S'})_* \mu_S = \mu_{S'}$.

2. Let $(X_i : (\{1,2,3,4,5,6\}^{\mathbb{N}}, \mathcal{F}, \mathbb{P}) \to \{1,2,3,4,5,6\})_{i=1}^{\infty}$ be the process modeling the rolling of a "fair" die (with six faces, with faces numbered "1" to "6") infinitely many times, with the assumption that the i-th roll is independent of the j-th, for any $i \neq j$.

   (a) What is the probability that the die-rolls produce the face numbered "1" hundred times in a row (that is, consecutively)?

   (b) Define $\mathcal{F}_i$ to be the σ-algebra generated by the RVs $X_1, \ldots, X_i$. Define the function $\tau_1 : \{1,2,3,4,5,6\}^{\mathbb{N}} \to \mathbb{N}$ as
   $$\tau_1(\omega) := \min\{i \geq 1 \mid X_i(\omega) = 1\}.$$
   Show that $\{\tau_1 \leq n\}$ is $\mathcal{F}_n$-measurable.

   (c) Show that τ_1 is $\mathcal{F}$-measurable.

   (d) Compute the probability that $\tau_1 \equiv 1 \mod 2$.

   (e) What is the expected time when "1" first appears during a sequence of die-rolls (that is, what is the expectation of τ_1)?

   (f) Define inductively the k-th time that "1" appears;
   $$\tau_1^{(k)}(\omega) := \min\{n > \tau_1^{(k-1)}(\omega) \mid X_n(\omega) = 1\},$$
   where $\tau_1^{(1)} := \tau_1$. Determine the limit RV of $\tau_1^{(n)}/n$ if it converges in probability.

   (g) Define $\tau_e(\omega) = \min\{i \geq 1 \mid X_i(\omega) \equiv 0 \mod 2\}$. Show that $\omega \mapsto X_{\tau_e(\omega)}(\omega)$ is $\mathcal{F}$-measurable.

3. Let $(X_i)_i$ be $\mathbb{Z}$-valued i.i.d. RVs with law μ. Define $S_n = \sum_{i=1}^n X_i$, for each n. Define $\mathcal{S}_i$ to be the σ-algebra generated by the R.Vs $S_1, \ldots, S_i$ (is it the same as $\mathcal{F}_i$?). Show that for any $A \subset \mathbb{Z}$, $\mathbb{P}$-a.s $\omega \in \mathbb{Z}^{\mathbb{N}}$,

$$\mathbb{P}[\{S_{n+1} \in A\} \,|\, \mathcal{S}_n](\omega) = \mathbb{P}[\{S_{n+1} \in A\} \,|\, \sigma(S_n)](\omega),$$

with the relevant conditional probabilities defined as in the lecture of 8/8/25.

4. Let $(X)_{i=1}^{\infty}$ be the process in $\{H, T\}^{\mathbb{N}}$ associated to (countably) infinitely many independent tosses of a fair coin. For $\omega \in \{H, T\}^{\mathbb{N}}$, define $C_1(\omega) := 1/4$ if $X_1(\omega) = H$ and $C_1(\omega) := 3/4$ otherwise. Define C_i inductively as follows. Suppose C_j has been defined for $1 \leq j < i$. Suppose $C_{i-1}(\omega)$ is the midpoint of the interval $[k2^{-(i-1)}, (k+1)2^{-(i-1)}]$, for $0 \leq k \leq 2^{i-1}$, in the "dyadic decomposition" of $[0, 1]$ into intervals of length $2^{-(i-1)}$. Then $C_i(\omega) := k2^{-(i-1)} + 2^{-i}$ if $X_i(\omega) = H$ and $C_i(\omega) := k2^{-(i-1)} + 3 \cdot 2^{-i}$ otherwise. Show that $C_\infty := \lim_{i \to \infty} C_i$ exists almost surely (even surely?) and the law of C_∞ is given by the Lebesgue measure in $[0, 1]$.

5. (St. Petersburg game). Construct an $\mathbb{N}$-valued i.i.d. process $(X_i)_{i=1}^{\infty}$, with law μ for X_i, where $\mu(\{2^j\}) = 2^{-j}$. For each $n \in \mathbb{N}$ set $b_n := 2^{\lceil \log_2 n \rceil + \lceil \log_2 \log_2 n \rceil}$ (the symbol denotes the "smallest integer greater than" function).

   (a) Show that the hypothesis for the weak law for triangular arrays is satisfied for $((\overline{X}_{n,k})_{k=1}^n)_{n=1}^{\infty}$, where $\overline{X}_{n,k} = X_k \cdot \mathbb{1}_{\{|X_k| \leq b_n\}}$.

   (b) Apply the result to conclude $\frac{S_n}{n \log n} \to 1$ in $\mathbb{P}$.

6. (Borel-Cantelli).

   (a) Show that if $A_n \subset (\Omega, \mathcal{F}, \mathbb{P})$, for $n \geq 1$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then

   $$\mathbb{P}[\{\omega \mid \omega \in A_n, \text{ for infinitely many } n\}] = 0.$$

   (Check that $\sum_{n=1}^{\infty} \mathbb{1}_{A_n}$ is in $L_1(\mathbb{P})$, to conclude it is almost surely finite.)

   (b) (Well-approximable reals). Show that set of reals x such that there is $\epsilon > 0$, for which

   $$|x - pq^{-1}| \leq q^{-(2+\epsilon)},$$

   for infinitely many (reduced) rationals pq^{-1} has Lebesgue measure zero. (Note that it suffices to restrict to $[0, 1]$. Apply the statement above.)

7. (Convergence in probability vs. almost sure convergence).

   (a) Show that almost sure/everywhere convergence implies convergence in probability.

   (b) Give an example of a sequence of independent RVs that converge in probability but not almost surely. (For the interested, can you find an example of such a sequence that is i.i.d.?)

(c) Show that if a sequence $(X_i)_{i=1}^{\infty}$ of i.i.d. random variables converge in probability to a random variable X, then one can find a subsequence which converges to X almost surely.
(Construct a subsequence $(i_k)_{k=1}^{\infty}$ such that $\mathbb{P}[\{|X_{i_k} - X| > 2^{-k}\}] \leq 2^{-k}$; apply Borel-Cantelli to $A_k := \{|X_{i_k} - X| > 2^{-k}\}$.)

8. (Layer-cake decomposition). Show that:

(a) for $H(x) = \int_{-\infty}^{x} h(t)\, dt$, with $h \geq 0$ Lebesgue-summable, and $\mathbb{R}$-valued f measurable in the measure space $(\Omega, \mathcal{F}, \mu)$,

$$\int_{\Omega} H \circ f \, du = \int_{-\infty}^{\infty} h(t)\, \mu(\{f > t\})\, dt,$$

(b) for $f \geq 0$ measurable in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $p \geq 1$, show that

$$\sum_{k=1}^{\infty} k^{p-1} \mathbb{P}[\{f > k+1\}] \leq \frac{\mathbb{E}_{\mathbb{P}}[f^p]}{p} \leq \sum_{k=0}^{\infty} (k+1)^{p-1} \mathbb{P}[\{f > k\}].$$