Assignment 2

Probability Theory (M. Math.)

The assignment is due by **03/10/25**. You may use results proved in class.

1. Let λ_1 and λ_2 denote the uniform measures on $\{0,1\}$ and $\{0,1,2\}$ respectively. Construct a process $(X_i)_{i=1}^{\infty}$ of independent RVs such that X_i has law λ_1 for odd i and law λ_2 for even i.

   (a) Prove that S_n/n converges a.s (where S_n is as usual).

   (b) Prove that there exist sequences $(c_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ with $s_n \to \infty$ such that

   $$\frac{S_n - c_n}{s_n} \xrightarrow{n\to\infty} N(0,1) \quad \text{in distribution.} \tag{1}$$

2. Construct a process $(X_i)_{i=1}^{\infty}$ of independent RVs, each with absolute value of mean and variance bounded by two, such that an almost sure limit $\lim_n S_n/n$ does not exist but the conclusion of a CLT as in (1) with respect to appropriate sequences $(c_n)_n$ and $(s_n)_n$ with $s_n \to \infty$ holds.

3. Construct a process $(X_i)_{i=1}^{\infty}$ of independent increments (i.e independent RVs) such that X_i has law given by the probability λ_i, where

   $$\lambda_i(\{1/2 - 1/2^i\}) = 1/2 \quad \text{and} \quad \lambda_i(\{1/2 + 1/2^i\}) = 1/2.$$

   (a) Prove or disprove that

   $$\frac{S_n - n/2}{s_n} \xrightarrow{n\to\infty} N(0,1) \quad \text{in distribution,}$$

   for some sequence $(s_n)_n$ with $s_n \to \infty$.

   (b) Is there a non-zero function $f : \mathbb{R} \to \mathbb{R}$ such that $f(S_n - n/2)$ converges in distribution? If so, to what?

4. Suppose $(X_i)_{i=1}^{\infty}$ is i.i.d. with mean zero, variance one. Let Z be an RV such that $Z, X_1, X_2, \ldots$ are independent and Z has a finite exponential moment, that is there is $\lambda > 0$ such that $\mathbb{E}[\exp(\lambda Z)] < \infty$. Show that $Y_i = ZX_i$ form an identically distributed sequence of RVs with mean zero. What is the variance σ^2 of Y_i? Show that $\mathbb{E}[\prod_{j=1}^{n} Y_{i_j}] = \prod_{j=1}^{n} \mathbb{E}[Y_{i_j}]$ for any finite sequence $\{j_1, \cdots, j_n\}$. Does $\frac{S_n}{\sigma\sqrt{n}}$ converge in distribution to the standard normal (i.e. $N(0,1)$)? If not, to what?

5. Prove that convergence in probability implies convergence in distribution.