Practice problems

Probability Theory (M. Math.)

1. Exercise 3.3.17 from Durrett.

2. Let X be an $\mathbb{R}$-valued random variable in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Show the following.

   (a) If $\mathbb{E}[X|\{\emptyset, \Omega\}] = \mathbb{E}[X]$.

   (b) If $\sigma(X), \mathcal{G}$ are independent, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

   (c) $\mathbb{E}[X|\sigma(X^2)] = |X| \cdot \mathbb{E}[\mathbb{1}_{\{X>0\}} - \mathbb{1}_{\{X<0\}}|\sigma(X^2)]$.

   (d) If $\{X > 0\}$ and X^2 are independent, and $\mathbb{P}[\{X > 0\}] > 1/2$, then $\mathbb{E}[X|\sigma(X^2)] > 0$ a.s.

   (e) If X has a continuous probability density function f, that is $\mathbb{P}[X \in (a, b)] = \int_a^b f(x)dx$, then

   $$\mathbb{E}[\mathbb{1}_{(0,\infty)} \circ X|\sigma(X^2)](\omega) = \frac{f(|X|(\omega))}{f(|X|(\omega)) + f(-|X|(\omega))}, \quad a.s.$$

   (Write $\mathbb{E}[\mathbb{1}_{(0,\infty)}|\sigma(X^2)] = h \circ X^2$, where h is Lebesgue measurable. Check that h may be taken to equal $\frac{f(\sqrt{z})}{f(\sqrt{z})+f(-\sqrt{z}))}$ at z, whenever the denominator is non-zero.)

   (f) Obtain a formula for $\mathbb{E}[X|\sigma(X^2)]$ in terms of f, given X has continuous density f.