Practice problems 2

Probability Theory (M. Math.)

1. If $g : (\Omega, \mathcal{F}, \mu) \to (R, \text{Lebesgue})$ is X-measurable and integrable, where

$$X : (\Omega, \mathcal{F}, \mu) \to (E, \mathcal{E}),$$

   then $g = \psi \circ X$, where $\psi : (E, \mathcal{E}) \to \mathbb{R}$ is Lebesgue measurable and $X_*\mathbb{P}$ integrable.

2. Use MCT to show that if λ is a stationary measure for a Markov chain with transition probabilities $(P_x)_{x \in S}$ in state space S, then $\mathbb{P}_\lambda$ is an invariant measure for the shift map T, that is, $T_*\mathbb{P}_\lambda = \mathbb{P}_\lambda$.

3. Let λ be an initial distribution. Show that functions which are products of characteristic functions, namely, $\mathbb{1}_{A_0} \circ Z_0 \cdots \mathbb{1}_{A_k} \circ Z_k$, for $k \in \mathbb{N}_0$ and $A_0, \ldots, A_k$ Borel, are contained in a monotone class of bounded Borel functions which satisfy

$$\mathbb{E}_{\mathbb{P}_\lambda}[g \circ T^j | \mathcal{F}_j](\omega) = \mathbb{E}_{Z_j(\omega)}[g]. \tag{1}$$

   Conclude that the equality holds almost surely for any bounded Borel measurable g.

4. Verify the Kolmogorov-Chapman equality

$$P_x^{(i+j)}(A) = \int_S P_y^{(i)}(A)\, dP_x^{(j)}(y),$$

   for all $A \subset S$ Borel and $j, i \in \mathbb{N}_0$, directly from (1). (Take $g = \mathbb{1}_A \circ Z_i$ and integrate both sides with suitable choice of λ.)