Practice problems 3

Probability Theory (M. Math.)

1. Consider the Markov chain with state space $\mathbb{R}$, transition probability $(P_x)_{x\in\mathbb{R}}$, where P_x is the distribution of a normal random variable with mean x and variance one, and initial distribution δ_0. Compute the conditional characteristic function $\mathbb{E}_0[e^{itX_{j+1}}|\sigma(X_j)]$, for $j \geq 1$.

2. (Stationary measures) Let S be a Polish space. Let $P = (P_x)_{x\in S}$ be a transition probability for a Markov chain.

   (a) If P has a stationary distribution then given $K \subset S$ compact, and $n \in \mathbb{N}$,

   $$\sup_{x\in S} P_x^{(n)}(K) = \sup_{x\in S} \mathbb{P}_x[\{Z_n \in K\}] > c_K,$$

   for some $c_K > 0$.

   (b) Let P be the transition density of the simple random walk in $\mathbb{Z}$, that is $P_x = \frac{1}{2}(\delta_{x-1}+\delta_{x+1})$. Show that if $(Z_n)_{n=0}^{\infty}$ is the simple random walk with $Z_0 = 0$ almost surely, then for $i \geq 1$, Z_i has the same distribution as the sum of i independent random variables with law $\frac{1}{2}(\delta_{-1} + \delta_1)$.

   (c) Use the inequality $P_x^{(n)}(y) \leq Cn^{-1/2}$, which holds for some $C > 0$ and all $n \in \mathbb{N}$, to show that the simple random walk has no stationary distribution.

   (d) Show for the simple random walk that $(\mathbb{Z}^{\mathbb{N}_0}, \mathbb{P}_\lambda, T)$ is a measure preserving system, where λ is the counting measure, and T is the left shift.

   (e) Let $x \in S$ be such that for any $\epsilon > 0$, there exists $K_\epsilon \subset S$ compact such that $P_x^{(n)}(K_\epsilon^c) < \epsilon$, for all $n \in \mathbb{N}$. Show that P has a stationary measure. (Use Prokhorov's theorem: if in a Polish space $(\mu_n)_{n=1}^{\infty}$ is a sequence of probability measures such that for any $\epsilon > 0$, there exists K_ϵ compact with $\mu_n(K_\epsilon^c) < \epsilon$, then μ_n weakly converges to a probability measure.)

   (f) Conclude that if S is compact then a stationary distribution always exists.

   (g) Show without using the inequality of Problem (2c) that the simple random walk in $\mathbb{Z}$ can not have a stationary probability.

(h) (Inward drift) Let $c > 0$. Consider the Markov chain in $\mathbb{Z}$ with transition probability

$$P_x = c\delta_0 + \frac{1-c}{2}(\delta_{x-1} + \delta_{x+1}) \quad \text{for } x \in S.$$

Show that P has a stationary probability, find it.

(i) Let $\#S < \infty$ and P be irreducible and aperiodic. Use Gelfand's formula $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ to conclude that $(P^n)_{xy} \xrightarrow{n\to\infty} \pi(y)$ exponentially. (Apply the formula to $P|_L$.)

(j) Let $\#S < \infty$ and P be irreducible and aperiodic. Show that π is unique: if π' is another stationary measure, then $0 \neq \pi' - \pi \in L$.

3. (Invariant sigma algebra) Let $(X, T, \mathcal{F}, \mu)$ be a probability preserving dynamical system. Set

$$\mathcal{I}_T = \sigma(\{A \mid T^{-1}A = A\}).$$

(a) Show that for $\varphi \in L^2(X, \mathcal{F}, \mu)$,

$$\mathbb{E}_\mu[\varphi \circ (Id - T) | \mathcal{I}_T] = 0.$$

Conclude that

$$C := \{g - g \circ T \mid g \in L^2(X, \mathcal{F}, \mu)\} \subset \mathrm{Ker}(L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{I}_T, \mu)),$$

the latter kernel of the projection.

(b) Show that if $f \perp \overline{C}$, then $f \in L^2(X, \mathcal{I}_T, \mu)$ (compute $\|f - f \circ T\|_{L^2}$). Conclude that

$$L^2(X, \mathcal{F}, \mu) = L^2(X, \mathcal{I}_T, \mu) \oplus \overline{C}.$$

4. Let S be countable and $(P_{xy})_{x\in S, y\in S}$ be a symmetric stochastic matrix. Show the following.

(a) $(P^{2n})_{xx} > 0$ for all $x \in S$, $n \in \mathbb{N}$.

(b) $\sup_{y,z\in S}(P^{2n})_{yz} \leq \sup_{x\in S}(P^{2n})_{xx}$.

(c) $\sup_{y,z\in S}(P^{2n+1})_{yz} \leq \sup_{x\in S}(P^{2n})_{xx}$.

5. Give an example of an irreducible aperiodic P on a finite state space S such that $vP \in \partial\Delta$, where $v \in \Delta$, where $\Delta = \{(x_1, \ldots, x_{\#S}) \mid \sum_{i=1}^{\#S} x_i = 1, x_j \geq 0 \,\forall j\}$ is the set of probability measures on S.

6. Let $\#S < \infty$ and P irreducible and aperiodic. Show that $\lim_{n\to\infty} \mathbb{E}_x[\exp(itZ_n)]$ exists for each $x \in S$. What is the limit? Does the limit as a function of x determine the stationary probability?