# Practice problems 4

## Probability Theory (M. Math.)

1. Show that if $S$ is countable, $P$ is irreducible and $y$ is recurrent, then
$$F_{xy} := \mathbb{P}_x[\tau_y < \infty] = 1,$$
where $\tau_y(\omega) = \inf\{k \geq 1 \mid Z_k(\omega) = y\}$, through the following steps.
   (a) Show that if $y$ is recurrent, then $z$ is recurrent for any $z \in S$.
   (b) Show that for all $k \in \mathbb{N}$,
   $$\mathbb{P}_y[\tau_y^{(k+1)} < \infty] = 1.$$
   Use the equality relating it to $\mathbb{P}_y[\tau_y^{(k)} < \infty]$; here $\tau_y^{(k+1)} = \inf\{n > \tau_y^{(k)} \mid Z_n = y\}$ with the convention that infimum of the empty set is infinity; $\tau_y^{(1)} = \tau_y$, $\tau_y^{(0)}$=0.
   (c) Show that
   $$\mathbb{P}_y[\tau_y^{(k+1)} < \infty] \leq \sum_{z \in S} (P^k)_{yz} \, \mathbb{P}_z[\tau_y < \infty].$$
   (d) Conclude that for any $y, z \in S$, $\mathbb{P}_z[\tau_y < \infty] = 1$.
2. Show that $(\tau_y^{(k+1)} - \tau_y^{(k)})_{k \geq 0}$ are iid with law $(\tau_y)_* \mathbb{P}_y$. (Check that $\tau_y^{(k+1)} - \tau_y^{(k)} = \tau_y \circ T^{\tau_y^{(k)}}$.)
3. Use MClassT to show that
$$\mathbb{E}_{\mathbb{P}}[fg|\mathcal{G}] = g\mathbb{E}_{\mathbb{P}}[f|\mathcal{G}],$$
for $\mathcal{G} \subset \mathcal{F}$, for $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $g$ bounded $\mathcal{G}$-measurable.