Brownian Motion and Diffusions

Homework 1

26 January 2026

Homework are for your own practice and are not meant to be turned in.

1. Show that a stochastic process $(X_t : t \geq 0)$ with continuous paths is a Brownian motion if and only if it is a Gaussian process with

$$\mathrm{E}(X_t) = 0\,, t \geq 0\,,$$

and

$$\mathrm{Cov}(X_s, X_t) = s \wedge t\,, s, t \geq 0\,.$$

2. Suppose that $(X_t : t \in I)$ is a Gaussian process. For $T_0, T_1 \subset I$, show that $(X_t : t \in T_0)$ and $(X_t : t \in T_1)$ are independent families if and only if

$$\mathrm{Cov}(X_s, X_t) = 0 \text{ for all } s \in T_0, t \in T_1\,.$$

3. Let $(B_t : t \geq 0)$ be a Brownian motion. Fix $s > 0$ and define

$$W_t = B_{s+t} - B_s\,, t \geq 0\,.$$

   (a) Show that $(W_t : t \geq 0)$ is also a Brownian motion.
   (b) Show that $(W_t : t \geq 0)$ is independent of $(B_t : 0 \leq t \leq s)$.
   (c) Prove that for $t > s$,

$$\mathrm{E}\left(B_t | \sigma(B_u : 0 \leq u \leq s)\right) = B_s\,.$$

4. Suppose T is an arbitrary set, and for every $k \geq 1$ and **distinct** $t_1, \ldots, t_k \in T$, there is a probability measure $\mu_{t_1,\ldots,t_k}$ on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. If the family $\{\mu_{t_1,\ldots,t_k} : k \geq 1, t_1, \ldots, t_k \in T \text{ distinct}\}$ satisfies the consistency criteria in the hypotheses of the Kolmogorov existence theorem, then show the following. Fix $k \geq 1$ and a permutation π of $(1, \ldots, k)$. Define the map ψ_π from $\mathbb{R}^k$ to $\mathbb{R}^k$ by

$$\psi_\pi(x_1, \ldots, x_k) = (x_{\pi 1}, \ldots, x_{\pi k})\,.$$

Show that for any $t_1, \ldots, t_k \in T$,

$$\mu_{t_1 \ldots t_k} = \mu_{t_{\pi 1} \ldots t_{\pi k}} \psi_\pi\,.$$

5. Suppose that Ξ is a field on a non-empty set Σ. If

$$\nu : \Xi \to [0, \infty]$$

is a function satisfying

$$\nu(A \cup B) = \nu(A) + \nu(B)\,, \text{ for all disjoint } A, B \in \Xi\,,$$

then show the following.

(a) If $A, B \in \Xi$ and $A \subset B$, then

$$\nu(A) \le \nu(B)\,.$$

(b) The set-function ν is finitely sub-additive, that is, for all $n \ge 1$,

$$\nu\left(\bigcup_{i=1}^n A_i\right) \le \sum_{i=1}^n \nu(A_i)\,, A_1, \ldots, A_n \in \Xi\,.$$

(c) Assuming, in addition, that $A_n \in \Xi$ and $A_n \downarrow \emptyset$ imply that

$$\lim_{n\to\infty} \nu(A_n) = 0\,,$$

show that ν is countably additive on Ξ.

6. Let

$$\Omega = \mathbb{R}^{[0,1]}\,,$$

that is, the set of all functions from $[0, 1]$ to $\mathbb{R}$, equipped with the sigma-field $\mathcal{A}$ which makes each one-dimensional projection measurable, as defined in the class. Show that

$$C[0, 1] \notin \mathcal{A}\,.$$

Hint: If Σ is a non-empty set, and $\mathcal{F}$ is a collection of functions from Σ to $\mathbb{R}$, then show that

$$\sigma(\mathcal{F}) = \bigcup_{\{f_1, f_2, \ldots\} \subset \mathcal{F}} \sigma(f_1, f_2, \ldots)\,.$$

7. Consider a set T, and functions

$$\mu : T \to \mathbb{R}\,,$$

and

$$R : T \times T \to \mathbb{R}\,,$$

such that

$$R(s, t) = R(t, s)\,, s, t \in T\,,$$

and for any $k \geq 1$ and $t_1, \ldots, t_k \in T$, the $k \times k$ matrix $((R(t_i, t_j) : 1 \leq i, j \leq k))$ is n.n.d. Show that there exists a stochastic process $(X_t : t \in \mathbb{R})$ such that for any $k \geq 1$ and $t_1, \ldots, t_k \in T$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ follows multivariate normal, and

$$\begin{aligned} \mathrm{E}(X_t) =& \mu(t)\,, t \in T\,, \\ \mathrm{Cov}(X_s, X_t) =& R(s,t)\,, s, t \in T\,. \end{aligned}$$

8. Given a finite measure μ on $\mathbb{R}$, which is symmetric about 0, show that there exists a stationary Gaussian process $(X_t : t \in \mathbb{R})$ with

$$\mathrm{E}(X_t) = 0\,, t \in \mathbb{R}$$

and

$$\mathrm{E}(X_s X_t) = \int_{-\infty}^{\infty} e^{\iota(t-s)x} \mu(dx)\,, s, t \in \mathbb{R}\,,$$

where $\iota = \sqrt{-1}$.

9. Let μ be as in the above exercise, and set

$$R(t) = \int_{-\infty}^{\infty} e^{\iota t x} \mu(dx)\,, t \in \mathbb{R}\,.$$

If

$$R(t) = \mu(\mathbb{R}) + O\left(|t|^{\varepsilon}\right)\,, t \to 0\,,$$

for some $\varepsilon > 0$, then show that (X_t) can be chosen to have continuous paths in the above exercise.

10. Suppose that $(B_t : t \geq 0)$ is a Brownian motion. Fix $c > 0$ and define

$$X_t = c^{-1/2} B_{ct}\,, t \geq 0\,.$$

Show that $(X_t : t \geq 0)$ is also a Brownian motion. In other words, if time is multiplied by c and space is scaled by $\sqrt{c}$, then a Brownian motion remains a Brownian motion. This property is known as *self-similarity*.

11. Let $(B_t : t \geq 0)$ be a Brownian motion. Define

$$X_t = \begin{cases} tB_{1/t}\,, & t > 0\,, \\ 0, & t = 0\,. \end{cases}$$

Show that $(X_t : t \geq 0)$ is a Brownian motion. This is the so-called *time inversion* property.

12. Let $(B_t : t \geq 0)$ be a Brownian motion. Show that for a fixed $t \geq 0$,

$$\lim_{h \to 0} \frac{B_{t+h} - B_t}{h}$$

does not exist a.s.

13. Show that there exists a stationary Gaussian process $(X_t : t \in \mathbb{R})$ with mean zero and **continuous paths** satisfying

$$\mathrm{E}(X_0 X_t) = e^{-|t|}\, , t \in \mathbb{R}\, .$$

This is the so-called *Ornstein-Uhlenbeck process*.

14. Let $(B_t : t \geq 0)$ be a Brownian motion. Set

$$X_t = e^{-t/2} B_{e^t}\, , t \in \mathbb{R}\, .$$

Show that $(X_t : t \geq 0)$ is a stationary process. Can you identify it?

15. Let $\mathbb{Q}_d$ be the set of dyadic rationals in $\mathbb{R}$. Suppose that $f : \mathbb{Q}_d \to \mathbb{R}$ is an uniformly continuous function, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| \leq \varepsilon \text{ for all } x, y \in \mathbb{Q}_d \text{ with } |x - y| \leq \delta\, .$$

Show that there exists an uniformly continuous function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = f(x)\, , x \in \mathbb{Q}_d\, .$$

Hint. If $\{x_n : n \geq 1\}$ is a Cauchy sequence in $\mathbb{Q}_d$, argue that $\{f(x_n)\}$ is a Cauchy sequence as well.