Lecture Notes 1 (Recap of ODEs)

Course: PDE (M. Math.)
Teacher: Debanjan Nandi

January 19, 2026

1. Introduction

- Let V be a space with differentiable structure (an Euclidean domain or a differentiable manifold).
- Let $f : V \to TV$ be a function in $C^0(V, TV)$ or $C^k(V, TV)$, for some $k \geq 1$.

The qualitative theory of ODE begins with the question of existence, dependence on initial conditions, and local and global behavior of solutions to equations of the form

$$\frac{d}{ds}\Big|_{s=t} \gamma(s) = f(\gamma(t)), \quad \forall t \in I, \tag{1}$$

where I is an (open, closed, partially open, or partially closed) interval, and γ is required to be a map $\gamma : I \to V$.

Remark: When considering a *non-autonomous* ODE, that is when the vector field changes with time; $t \to f_t$,

$$\frac{d}{ds}\Big|_{s=t} \gamma(s) = f(t, \gamma(t)), \tag{2}$$

then writing $\hat{\gamma}(t) = (t, \gamma(t))$, a solution γ to (2) exists, if and only if

$$\frac{d}{ds}\Big|_{s=t} \hat{\gamma}(s) = \hat{f}(\hat{\gamma}(t)),$$

where $\hat{f}(y) = (1, f(y))$ (or $(1, f(y)^\intercal)^\intercal$ if your vectors are column vectors), which is an equation of type (1). An ODE of type (1) is called *autonomous*.

2. Local existence and uniqueness

(a) Let $V \subset \mathbb{R}^n$ be open. Let $f : V \to \mathbb{R}^n$ be locally Lipschitz. That is for all $K \subset V$ precompact, there is $L_K > 0$ such that

$$\|f(x) - f(y)\| \leq L_K \|x - y\|, \quad \forall\, x, y \in K.$$

Consider the IVP (initial value problem) where one asks for a solution of (1) subject to the condition $\gamma(t_0) = x_0$, where $t_0 \in \mathbb{R}$ and $x_0 \in V$. The local existence problem asks if there is an interval I containing t_0 such that a map $\gamma : I \to V$ exists, satisfying (1) and $\gamma(t_0) = x_0$.
Show that it is enough to show existence for $t_0 = 0$.
Picard's approach to the question of local existence begins by considering the integral form of the ODE which would be an implicit formula for the solution being sought:

$$\gamma(t) = \gamma(0) + \int_0^t f(\gamma(s))\, ds, \tag{3}$$

where the integral is evaluated coordinatewise. A γ solving the IVP exists in some I containing zero, iff a continuous γ satisfying (3) for $t \in I$ exists (**show** this).

(b) (Existence)

i. Picard defines successive approximations (Picard iteration)

$$\gamma_k(t) = x_0 + \int_0^t f(\gamma_{k-1}(s))\,ds; \quad \gamma_0 \equiv x_0.$$

ii. As discussed in class, for any $\delta > 0$ such that $\overline{B}(x_0, \delta) \subset V$, there is $\epsilon > 0$ such that for $t \in [-\epsilon, \epsilon]$, $\gamma_k(t) \in \overline{B}(x_0, \delta)$, for $k \geq 0$. This uses only continuity of f.

iii. Banach's viewpoint: Consider the (nonlinear) mapping

$$C^0([-\epsilon, \epsilon], \overline{B}(x_0, \delta)) \ni \gamma \mapsto K(\gamma), \quad K(\gamma)(t) = x_0 + \int_0^t f(\gamma(s))\,ds.$$

iv. Using the Lipschitz property of f in $\overline{B}(x_0, \delta)$, $\epsilon > 0$ may be chosen small enough that γ_k converge uniformly on $[-\epsilon, \epsilon]$ and K is a contraction mapping. This leads to a solution/fixed point.

(c) (Uniqueness) As discussed in class, if γ_1, γ_2 both solve the IVP, then it can be shown using (3) that $\|\gamma_1 - \gamma_2\| \leq \|\gamma_1 - \gamma_2\|/2$ in the sup-norm, for $\epsilon > 0$ small enough.

(d) (Peano and dropping Lipschitz) This has been discussed in class. I might add something here later. Existence holds under the assumption of continuity alone, but uniqueness fails as examples show (see below).

3. **Dependence on initial conditions**

Earlier we looked for solutions from an interval to the space satisfying (1). Now we will consider a bunch of initial conditions simultaneously and study

- the regularity of the solutions/trajectories and
- how they diverge at future times with respect to distance between initial points.

For that we will look not for existence of solutions as just curves solving the ODE, but as maps which carry the initial data as input and solve the ODE with respect to the time parameter for each initial condition. The result is as follows.

Let $V \subset \mathbb{R}^n$ be open. Let $f : V \to \mathbb{R}^n$ be locally Lipschitz. Then for any $x_0 \in V$, there are $\delta = \delta(f) > 0$ and $\epsilon = \epsilon(f) > 0$ for which there is a mapping $\phi : [-\epsilon, \epsilon] \times B(x_0, \delta) \to V$ such that

$$\frac{d}{ds}\phi(t, x) = f(\phi(t, x)), \quad \phi(0, x) = x.$$

Moreover,

1. $\|\phi(t, x) - \phi(t, x')\| \leq \|x - x'\|e^{Lt}$, for any $x, x' \in B(x_0, \delta)$ and $L = L(f) > 0$.
2. $t \mapsto \phi(t, x)$ is continuously differentiable for each $x \in B(x_0, \delta)$,
3. if $f \in C^1(V, \mathbb{R}^n)$, then $t \mapsto \phi(t, x)$ is twice continuously differentiable for each $x \in B(x_0, \delta)$,
4. if $f \in C^1(V, \mathbb{R}^n)$, then $\phi \in C^1([-\epsilon, \epsilon] \times B(x_0, \delta), V)$, that is the dependence on data is even smooth.

We outline a proof here. For details, either fill them in, or see Perko.

(a) Recalling the computation in Picard's iteration, note that for any $x_0 \in V$, we had an $\epsilon > 0$ depending on the Lipschitz constant of f in a neighborhood of x_0 such that a solution to the IVP with $\gamma(0) = x$ exists in $[-\epsilon, \epsilon]$. The same $\epsilon > 0$ actually works for all initial points in a small enough neighborhood of x. More precisely:

i. For each $x \in V$ define Picard approximants for $\phi_k(t,x)$ formally;

$$\forall\, k \geq 1,\;\; \phi_k(t,x) = x + \int_0^t f(\phi_{k-1}(s,x))\, ds, \quad \text{and} \quad \phi_0(\cdot,x) \equiv x.$$

ii. **Show** using the local Lipschitz continuity of f, to get a recursive bound between the distance between consecutive approximants, that if we restrict x to a precompact set U, then for any $\delta > 0$ small, taking $\epsilon = M_U\delta/L_U^2$, $\phi_k(t,x) \in B(U, 2M_U\delta/L_U)$, where $M_U = \sup_{x\in U}\|f\|$, L_U is the local Lipschitz constant, for all $x \in U$, $k \geq 0$, $t \in [-\epsilon,\epsilon]$.

iii. Then using a triangle inequality argument **show** that there is $\epsilon > 0$ $\delta > 0$, depending on x_0, f, such that for all $t \in [-\epsilon,\epsilon]$, $x \in B$ $\phi_k(t,x) \in \overline{B}(x_0,\delta/2)$.

iv. **Show** using the fact that the approximants converge exponentially fast uniformly for each $(t,x) \in [-\epsilon,\epsilon] \times \overline{B}(x_0,\delta/4)$, as $k \to \infty$, using the Lipschitz property of f (exactly as in the existence theorem, by recursion).

v. Denote the pointwise limit by ϕ, and **show** that it satisfies

$$\phi(t,x) = x + \int_0^t f(\phi(s,x))\, ds. \tag{4}$$

(b) Now we discuss the regularity aspects.

- We hope to extract regularity information from the integral inequality (4), which is an explicit formula. Irregularity at smaller times than t may get fed into the behavior at time t and add up.
- We need a tool to check this is not the case and in fact the solution is at least as smooth as f. This tool is precisely Gronwall's inequality.

Gronwall's inequaity If $C_1, C_2 > 0$ are such that for a function $0 \leq g \in L^1([0,a],\mathbb{R})$,

$$g(t) \leq C_1 + C_2 \int_0^t g(s)\, ds, \quad \forall\, t \in [0,a],$$

we have

$$g(t) \leq C_1 e^{C_2 t}, \quad \forall\, t \in [0,a].$$

i. The claims involving time derivatives are simple observations from (4).

ii. Using (4), for $(t,x) \in [-\epsilon,\epsilon] \times \overline{B}(x_0,\delta/4)$,

$$\|\phi(t,x) - \phi(t,x')\| \leq \|x - x'\| + L\int_0^t \|\phi(s,x) - \phi(s,x')\|\, ds,$$

where L is the local Lipschitz constant.

iii. Applying Gronwall's inequality (see below), we get for $(t,x) \in [-\epsilon,\epsilon] \times \overline{B}(x_0,\delta/4)$,

$$\|\phi(t,x) - \phi(t,x')\| \leq \|x - x'\| e^{Lt}. \tag{5}$$

iv. For f with a continuous Df, one may expect for each $t \in [-\epsilon,\epsilon]$, $D_x\phi(t,x)$ (that is the total derivative of $(x \mapsto \phi(t,x))$ to exist and be continuous. Suppose it does exist. Then it should satisfy

$$\left.\frac{d}{ds}\right|_{s=t} D_x\phi(t,x) = D_x \left.\frac{d}{ds}\right|_{s=t}\phi(t,x) = D_x f(\phi(t,x)) = Df(\phi(t,x)) \circ D_x\phi(t,x), \tag{6}$$

$$D_x\phi(0,x) = id.$$

Set up a Picard type iteration to **show** that if $(t,x) \to A(t,x) \in M_{n\times n}(\mathbb{R})$ is continuous in $[-\epsilon_0,\epsilon_0] \times V$, then there is $0 < \epsilon < \epsilon_0$ such that for any open U precompact, such that a solution of

$$\dot{\theta}(t,x) = A(t,x)\theta(t,x), \quad \theta(0,x) = id$$

exists in $[-\epsilon,\epsilon]$, and $\theta \in C^0([-\epsilon,\epsilon] \times U)$.

v. Let Φ be a solution of (6). It follows that for $h \in \mathbb{R}^n$, with $\|h\|$ small,

$$\|\phi(t,x+h) - \phi(t,x) - \Phi(t,x)h\| \leq$$
$$\int_0^t \|f(\phi(s,x+h)) - f(\phi(s,x)) - Df(\phi(s,x))(\phi(s+h,x) - \phi(s,x))\|\, ds$$
$$+ \int_0^t \|Df(\phi(s,x))\|\|\phi(s,x+h) - \phi(s,x) - \Phi(s,x)h\|\, ds$$

Using differentiability of f (which leads to an application of (8)) followed by another application of Gronwall's inequality implies $D_x\phi(t,x)$ exists and is continuous in (t,x).

4. **Recall of some basic notions from differentiable geometry**

We discuss some notions relevant for the course.

(a) (Manifold)

A C^k-manifold V, $k \geq 1$, is a second countable, locally compact topological space with data $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in \mathcal{J}\}$ such that $U_\alpha \subset V$ are open for $\alpha \in \mathcal{J}$, $V = \cup_{\alpha\in\mathcal{J}} U_\alpha$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is homeo for $\alpha \in \mathcal{J}$, and for each $\alpha, \beta \in \mathcal{J}$,

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \in C^k,$$

as a mapping of Euclidean domains. For us always $k \geq 2$. The $(U_\alpha, \varphi_\alpha)$ are called *charts*. A collection $((U_\alpha, \varphi_\alpha))_\alpha$ satisfying the properties above is called an *atlas*. The $\varphi_\alpha \circ \varphi_\beta^{-1}$ are called *transition functions*. Here n is the *dimension* of V.

(b) (Smooth map)

Let V be as above. And let W be another C^k-manifold with data $\{(R_\beta, \psi_\beta) | \beta \in \mathcal{I}\}$. A mapping $f : V \to W$ is in $C^r(V, W)$ for some $0 \leq r \leq k$, if for any $\alpha \in \mathcal{J}, \beta \in \mathcal{I}$,

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} \in C^r(\varphi_\alpha(U_\alpha \cap f^{-1}(R_\beta)), \mathbb{R}^n).$$

(c) (Smooth curve)

Note that any open interval I is a smooth (that is C^∞) manifold with data (I, id). A C^r-curve γ is a map $\gamma : I \to V \in C^r(I, V)$.

(d) (Smooth function)

A C^r-function f is a map $f \in C^r(V, \mathbb{R})$.

(e) (Tangent space)

For V as above, and $x \in V$, any C^1-curve $\gamma : (-\epsilon, \epsilon) \to V$, with $\gamma(0) = x$, defines a tangent vector written as v_γ, or $\dot{(\gamma)}(0)$, at x. This tangent vector is a linear functional on the algebra $C^1(V, \mathbb{R})$, defined by

$$f \mapsto \frac{d}{dt}\Big|_{t=0} f \circ \gamma.$$

Note that trivially, U_α is a C^∞ manifold (with chart (U_α, id)) and if $x \in U_\alpha$, we can consider for the same γ the functional $w_\gamma : C^1(\varphi_\alpha(U_\alpha), \mathbb{R}) \to \mathbb{R}$ sending

$$g \mapsto \frac{d}{dt}\Big|_{t=0} g \circ \varphi_\alpha \circ \gamma = \sum_{i=1}^n \frac{d(\varphi_\alpha)_i \circ \gamma}{dt} \frac{\partial g}{\partial x_i}$$

This is the directional derivative of the function g along the vector $\left(\frac{d(\varphi_\alpha)_1 \circ \gamma}{dt}, \cdots, \frac{d(\varphi_\alpha)_n \circ \gamma}{dt}\right)$; here $(\varphi_\alpha)_i$ are the coordinate projections composed with φ_α. Note that any direction/vector can be written in this form for some curve γ. And so the space of directions at $\varphi_\alpha(x)$ along which to differentiate any g at $\varphi_\alpha(x)$, or the tangent space at $\varphi_\alpha(x)$, is identified with functionals $\sum_{i=1}^n \frac{d(\varphi_\alpha)_i \circ \gamma}{dt} \frac{\partial}{\partial x_i}$, which form a vector space spanned by the functionals $\frac{\partial}{\partial x_i}$. Thus under this identification the vector w_γ is

$$w_\gamma = \left(\frac{d((\varphi_\alpha)_1 \circ \gamma)}{dt}, \cdots, \frac{d((\varphi_\alpha)_n \circ \gamma)}{dt} \right),$$

and in its functional form it is

$$w_\gamma = \sum_{i=1}^n \frac{d((\varphi_\alpha)_i \circ \gamma)}{dt} \frac{\partial}{\partial x_i},$$

and acts on differentiable functions in open Euclidean set $\varphi_\alpha(U_\alpha)$, which is the same as above, but written as a linear combination of basis elements, instead of a tuple.

Now we define elements of tangent space at a point in V in exactly the same way. Note that the functionals v_γ (resp. w_γ) only require the values of the test functions in arbitrarily small neighborhoods of x (resp. $\varphi_\alpha(x)$). After taking the natural equivalence classes it is observed that $g \circ \varphi_\alpha \mapsto g$ gives a one to one correspondence between the algebras (of functions in V, C^1 in manifold sense, and functions in $\varphi_\alpha(U_\alpha)$, C^1 in Euclidean sense) where v_γ and w_γ act on; and moreover, under this correspondence we have

$$v_\gamma(g \circ \varphi_\alpha) = w_\gamma(g) =: (\varphi_\alpha)_* v_\gamma(g).$$

The correspondence $v_\gamma \mapsto w_\gamma = (\varphi_\alpha)_* v_\gamma$ is one to one and onto.

So the tangent space at x, defined

$$TV_x := \{v_\gamma \mid \gamma \in C^1(I_\gamma, V),\ \gamma(0) = x\}.$$

is the set of "directions to go from x" along which we compute directional derivatives of C^1 functions (and maps), with the linear functional aspect of a direction being the operation of taking directional derivatives of functions along it.

In summary:

- Any C_1 curve $\gamma : (-\epsilon, \epsilon) \to V$ with $\gamma(0) = x$ will determine an element of TV_x, written as $\dot\gamma(0)$ or $\frac{d}{dt}\Big|_{t=0} \gamma(t)$ and this vector is an element in the space of linear maps from $C^\infty(V, \mathbb{R}) \to \mathbb{R}$ acting as defined above, as directional derivatives.
- Via local coordinates we get a coordinate representation of $v_\gamma = \dot\gamma(0)$

$$(\varphi_\alpha)_*(\dot\gamma(0)) = \left(\frac{d}{ds}\Big|_{s=t} (\varphi_\alpha)_1 \circ \gamma(s), \cdots, \frac{d}{ds}\Big|_{s=t} (\varphi_\alpha)_n \circ \gamma(s) \right) = \sum_{i=1}^n \frac{d}{ds}\Big|_{s=t} (\varphi_\alpha)_i \circ \gamma(s) \frac{\partial}{\partial x_i},$$

  where $\frac{d}{ds}\Big|_{s=t} (\varphi_\alpha)_i \circ \gamma(s)$ are just derivatives of vector-valued functions of one variable in the Euclidean sense.
- It can be checked (by hand, see for yourself) that the only functionals in $T\varphi(U_\alpha)_{\varphi_\alpha(x)}$ which are derivations, or appear as time derivatives of smooth curves in $\varphi_\alpha(U_\alpha)$ are linear combinations of $\frac{\partial}{\partial x_i}$. Moreover, any such linear combination is the time derivative of a smooth curve in $\varphi_\alpha(U_\alpha)$ (just take a straight line pointed at the required direction)
- (Localization gives the identification) Since the space of smooth functions in the chart and in the manifold, quotiented with the above mentioned equivalence relation localizing at x, become isomorphic, we see that the functionals defined at the beginning must be in one-one correspondence with those in the tangent space at a point in the chart (space of linear combinations of $\frac{\partial}{\partial x_i}$ with real coefficients).
- So for every $v \in TV_x$, there is a smooth curve $\gamma : I \to V$ such that $0 \in I$, $\gamma(0) = x$ with $\dot\gamma(0) = v$ acting on functions as the directional derivative operator. And for for every C^1 curve $\gamma : I \to V$, with $\gamma(0) = x$, $\dot\gamma(0)$ is a vector in TV_x, that is directional derivative operator.

- If $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ are two vectors at x then the action of $\lambda\dot{\gamma}_1(0)+\mu\dot{\gamma}_2(0)$ on $C^1(V,\mathbb{R})$ is identical to the action of $\lambda(\varphi_\alpha)_*(\dot{\gamma}_1(0))+\mu(\varphi_\alpha)_*(\dot{\gamma}_2(0))$ on $C^1(\varphi_\alpha(U_\alpha),\mathbb{R})$. But the latter is a linear combination of $\frac{\partial}{\partial x_i}$'s. So we can find a smooth curve in $\varphi_\alpha(U_\alpha)$ whose derivative at zero is $\lambda(\varphi_\alpha)_*(\dot{\gamma}_1(0)) + \mu(\varphi_\alpha)_*(\dot{\gamma}_2(0))$. Pulling it back by postcomposing with φ_α^{-1} we get a curve $\gamma : I \to V$, such that $\dot{\gamma}(0) = \lambda\dot{\gamma}_1(0) + \mu\dot{\gamma}_2(0)$. So the tangent space TV_x contains linear combination of vectors in it and is a vector space of dimension n. This is another way of saying that $(\varphi_\alpha)_*$ which is a bijection, is linear.

Remark: This is the vector space of derivations on $C^r(V,\mathbb{R})$ (with the latter quotiented by the relation of local equivalence between functions, where two functions are equivalent if their values overlap on some open set containing x; the local $\mathbb{R}$-algebra at x). The vector space of derivations is also dual to $\mathfrak{m}_x/\mathfrak{m}_x^2$, the space of differentials of functions; here $\mathfrak{m}_x$ is the maximal ideal consisting of equivalence classes of functions (germs) that vanish at x. These interpretations will not however be useful for us.

(f) (Tangent bundle)
For V as above, the tangent bundle TV, as a set $\sqcup_{x\in V} TV_x$, itself gets a structure of a $2n$-dimensional differentiable manifold by first defining the topology by choosing as basis $\{\sqcup_{x\in U} TV_x \mid U \subset V \text{ open}\}$ and then attaching the data

$$\{(\sqcup_{x\in U_\alpha} TV_x, \hat{\varphi}_\alpha) \mid \alpha \in \mathcal{J}\},$$

where

$$\hat{\varphi}_\alpha(v) = (\varphi_\alpha(x), (\varphi_\alpha)_*(v)), \quad \forall v \in TV_x,\ \alpha \in \mathcal{J},$$

and $(\varphi_\alpha)_*$ is the map defined in (e). There is a natural surjection

$$\pi : TV \to V,$$

which is defined as

$$\pi(v) = x, \quad \forall\, v \in TV_x,\ x \in V.$$

Check for yourself that it is a smooth surjection.

(g) **Show** that $D(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ (\varphi_\beta)_* = (\varphi_\alpha)_*$, $\forall \alpha, \beta \in \mathcal{J}$, where $D(\varphi_\alpha \circ \varphi_\beta^{-1})$ is the total derivative of an Euclidean diffeomorphism. This can be used to show that the charts used for the tangent bundle indeed form an atlas.

(h) (Derivative) For a map $f : V \to W$, $Df(x)$ exists if

$$\forall\, g \in C^1(W,\mathbb{R}),\ \ \forall\, v_\gamma \in TV_x, \quad \frac{d}{dt}\Big|_{t=0} g \circ f \circ \gamma(t) \ \text{ exists.}$$

Then there is a linear map $Df(x) : TV_x \to TW_x$. If $f \in C^1(V,W)$ then $Df(x) : TV_x \to TW_{f(x)}$ exists for all $x \in V$.

(i) (Going from manifold to $\mathbb{R}^n$) Let V be a C^∞-manifold, and $\gamma : I \to V$ a smooth map, where I is an interval. Then we know that for any $t \in I$, $\dot{\gamma}(t) = \frac{d}{ds}\Big|_{s=t}\gamma(s) \in TV_{\gamma(t)}$. As a vector it $\dot{\gamma}(t)$ is defined by the property that for any $g \in C^\infty(V,\mathbb{R})$, we can compute the directional derivative of g along $\dot{\gamma}(t)$. It is precisely

$$Dg(\dot{\gamma}(t)) = \frac{d}{ds}\Big|_{s=t} g \circ \gamma(s).$$

We get a map from I to TV, defined

$$\dot{\gamma} : I \to TV, \quad t \mapsto \dot{\gamma}(t) \in TV_{\gamma(t)}.$$

Suppose now that we are given a C^1 vector field f in V. By definition this object is

$$f \in C^1(V,TV) \ \text{ such that } \ \pi \circ f = id$$

in V, that is, the vector part of $f(x)$ is an element of TV_x and the position part of $f(x)$ is x, for any $x \in V$.

Let $\gamma : I \to V$ be differentiable, such that

$$\frac{d}{ds}\Big|_{s=t} \gamma(s) = f(\gamma(t)).$$

Let us read this equation in terms of the local data, that is in a chart. Fix $t \in I$. Let (U, φ) be a chart such that $\varphi : U \to \mathbb{R}^n$ is a homeomorphism and $\gamma(t) \in U \subset V$.

The corresponding chart around $\dot{\gamma}(t) \in TV$ is $(\sqcup_{x\in U} TV_x, \hat{\varphi})$, and the evaluation of $\hat{\varphi}$ at the point $\gamma(t) \in TV_{\gamma(t)} \subset \sqcup_{x\in U} TV_x$ is

$$\begin{aligned}\hat{\varphi}(\dot{\gamma}(t)) &= (\varphi(\gamma(t)), \varphi_*(\dot{\gamma}(t))) \\ &= \left(\varphi_1 \circ \gamma(t), \cdots, \varphi_n \circ \gamma(t), \frac{d}{ds}\Big|_{s=t} (\varphi_1 \circ \gamma)(s), \cdots, \frac{d}{ds}\Big|_{s=t} (\varphi_n \circ \gamma)(s)\right)\end{aligned}$$

Note that already $f(\gamma(t)) \in TV_{\gamma(t)}$, by the property $\pi \circ f = id$. Its coordinate representation becomes

$$\hat{\varphi}(f(\gamma(t))) = (\varphi_1 \circ \gamma(t), \cdots, \varphi_n \circ \gamma(t), \varphi_*(f(\gamma(t)))) .$$

- So for all $t \in I$, such that $\gamma(t) \in U$ (this is an open set of times), $\dot{\gamma}(t) = f(\gamma(t))$ is equivalent to after applying $\hat{\varphi}$,

$$\begin{aligned}\frac{d}{ds}\Big|_{s=t} \varphi \circ \gamma(s) &= \left(\frac{d}{ds}\Big|_{s=t} (\varphi_1 \circ \gamma)(s), \cdots, \frac{d}{ds}\Big|_{s=t} (\varphi_n \circ \gamma)(s)\right) \\ &= \varphi_*(f(\gamma(t))) \\ &= \varphi_* \circ f \circ \varphi^{-1}(\varphi \circ \gamma).\end{aligned} \tag{7}$$

- Above is an equation relating vectors in $\mathbb{R}^n$. So the equation $\dot{\gamma}(t) = f(\gamma(t))$ in V, becomes equivalent (in the sense that equality holds in manifold if and only if equality holds in $\mathbb{R}^n$) to the following setup in Euclidean space. The domain is $\varphi(U) = \mathbb{R}^n$. The vector field is

$$\varphi_* \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n.$$

  The curve satisfying the differential equation (and the one we are usually looking to find) is $\varphi \circ \gamma$. Once (7) has been solved the required local solution in the manifold is obtained by postcomposing with φ^{-1}.
- If $V \subset \mathbb{R}^n$ open. Then a local data/chart is (V, id). The local data at TV is $(\sqcup_{x\in V} TV_x, \hat{id})$ which is given by

$$\hat{id}(v) = (x, v), \quad \forall\, v \in TV_x,\ x \in V,$$

  as $id_* = id$ (checked easily by definition). This gives in particular a homeomorphism from TV to $V \times \mathbb{R}^n$. This is in fact a smooth map in the sense of manifold maps. If $f : V \to TV = V \times \mathbb{R}^n$ is a vector field, then since $\pi \circ f = id$, $f(x) = (x, F(x))$, for some $F : V \to \mathbb{R}^n$. Also a map $\gamma : I \to TV$ in local charts for TV is $t \mapsto (\gamma(t), \dot{\gamma}(t))$. So in local coordinates, we are looking at the equation

$$\dot{\gamma}(t) = F(\gamma(t)), \quad \text{where} \quad F : V \to \mathbb{R}^n.$$

  Remark If $V \subset \mathbb{R}^n$ is open and $g : V \to \mathbb{R}$ is C^1, then $x \mapsto \nabla g(x)$ is a map from $V \to TV$ whose coordinate representation is $x \mapsto (x, \nabla g(x))$, by definitions above. In writing $x \mapsto \nabla g(x)$ in the manifold sense, we include the data that $\nabla g(x)$ is not just any vector in $\mathbb{R}^n$, but an element of TV_x. This implicit infomration becomes explicit when writing the coordinate representation of the map. So w.r.t the language of the previous point $f(x) = (x, \nabla g(x))$ is a vector field in local coordinates and $x \mapsto F(x) = \nabla g(x)$ is just a vector-valued map from V to $\mathbb{R}^n$.

(j) If f is a C^1-vector field, then $f \in C^1(V, TV)$ as above, which is the same as saying

$$Df(x) : TV_x \to TTV_{f(x)}$$

exists as a linear map as above, $\mathrm{Hom}_f(TV, TTV)$ is a topological manifold and $Df : V \to \mathrm{Hom}_f(TV, TTV)$ is continuous, that is, for all $g \in C^1(TV, \mathbb{R})$,

$$(x, v_\gamma) \mapsto \frac{d}{dt}\Big|_{t=0} g \circ f \circ \gamma(t) \tag{8}$$

is continuous.

Remark: Here $\mathrm{Hom}_f(TV, TTV)$ being a topological manifold means that the transition maps are only homeomorphisms, not necessarily differentiable. Some details are as follows.

(Derivative Map) In general if $f : V \to W$ is a continuous function, then as a set $\mathrm{Hom}_f(TV, TW)$ where W is a C^k m-dimensional manifold, is $V \times M_{n\times m}(\mathbb{R})$, with chart

$$\hat{\varphi}_\alpha(x, A) = (\varphi_\alpha(x), (\varphi_\alpha)_* A),$$

where

$$A \in \mathrm{Hom}(TV_x, TW_{f(x)}), \quad (\varphi_\alpha)_* A \in \mathrm{Hom}(T(U_\alpha)_{\varphi_\alpha(x)}, TW_{f(x)}), \quad \text{and} \quad (\varphi_\alpha)_* A(w_\gamma) = A(v_\gamma).$$

If f is C^k, then $\mathrm{Hom}_f(TV, TW)$ is C^{k-1} and $Df \in C^{k-1}(V, \mathrm{Hom}_f(TV, TW))$ (a section of the vector bundle, again $\pi \circ Df = id$, where $\pi : \mathrm{Hom}_f(V, TV) \to V$ is the projection constant on fibers, as before). Perhaps a more standard way to write the vector bundle $\mathrm{Hom}_f(TV, TW)$ in the literature is $T^*V \otimes f^*TW$, where f^* denotes the pullback. To see why the regularity drops note that the chart $\hat{\varphi}_\alpha$ includes the derivative. For us, it is enough to understand continuity using (8).

(k) (Local existence, uniqueness and dependence on initial conditions in the manifold setting)
Use 4.i. to translate the local theory in manifolds to the Euclidean setting and provide a general statement for local existence, uniqueness and regularity of solutions in the case of smooth manifolds.

5. **Examples**:

(a) ODEs in $\mathbb{S}^2$

i. Consider a C^1 function $g : \mathbb{S}^2 \to \mathbb{R}$. Consider the IVP

$$\dot{\gamma}(t) = \nabla g(\gamma(t)), \quad \gamma(0) = x_0.$$

- Let us choose a chart and write $x \mapsto \nabla g(x)$ in local coordinates with respect to the chart. Let $U_i^+ = \{x_i > 0\} \cap \mathbb{S}^2$, $U_i^- = \{x_i < 0\} \cap \mathbb{S}^2$, for $i \in \{1, 2, 3\}$. The local data is $((U_i^+, \pi_i), (U_i^-, \pi_i))_{i=1}^3$. Here π_i is the coordinate projection from $\mathbb{R}^3$ to $\{x_i = 0\}$ (the projections are homeomorphisms restricted to the domain). This is the usual local data on $\mathbb{S}^2$.
- Suppose $x_0 \in U_1^+$. We obtain the coordinate representation of the vector field ∇g with respect to the chart (U_1^+, π_1) as follows. For $\pi_1(x) \in \pi_1(U_1^+)$, the vectors e_2 and e_3 in their directional derivative operator form are $\frac{\partial}{\partial x_2}\Big|_{\pi_1(x)}$ and $\frac{\partial}{\partial x_3}\Big|_{\pi_1(x)}$ respectively, which provide the partial derivatives of a function differentiable function in $\pi_1(U_1^+)$ at the point $\pi_1(x)$.
- Recall the isomorphism $(\pi_1)_*$ mapping vectors in TV_x to their coordinate reps, which are vectors in $T\pi_1(U_1^+)_{\pi_1(x)}$. We have $\frac{\partial}{\partial x_i}\Big|_{\pi_1(x)} \in T\pi_1(U_1^+)_{\pi_1(x)}$, for $i = 2, 3$ and these two vectors/directional derivative-operators form a basis. **Show** that their preimages wrt $(\pi_1)_*$ in TV_x are obtained by differentiating the curves

$$t \mapsto \pi_1^{-1}(x + te_i), \quad \text{for} \quad i = 2, 3$$

as $\mathbb{R}^3$-valued maps of time (checked directly from definition).

- The vector $\nabla g(x)$ is defined by the following property.

$$\langle \nabla g(x), \dot{\gamma}(0)\rangle_{\mathbb{S}^2} = \dot{\gamma}(0)(g),$$

for any differentiable curve $\gamma : I \to \mathbb{S}^2$, with $\gamma(0) = x$, and where the inner production $\langle \cdot, \cdot \rangle_{\mathbb{S}^2}$ is the inner product at $T\mathbb{S}^2_x$ obtained by restricting the Euclidean inner product from $T\mathbb{R}^3_x$.

- **Compute** the following. Note that $\left((\pi_1)_*^{-1} \frac{\partial}{\partial x_2}\Big|_{\pi_1(x)}, (\pi_1)_*^{-1} \frac{\partial}{\partial x_3}\Big|_{\pi_1(x)} \right)$ form a basis of TV_x. Compute the coordinate-coefficients $A^2(x)$ and $A^3(x)$ of $\nabla g(x)$ wrt this basis. Then **show** that the vector field $(\pi_1)_* \circ \nabla g \circ \pi_1^{-1}$ in $\pi_1(U_1^+)$ is given by

$$z \mapsto A_2(\pi_1^{-1}(z)) \frac{\partial}{\partial x_2}\Big|_z + A_3(\pi_1^{-1}(z)) \frac{\partial}{\partial x_3}\Big|_z.$$

- Use 4.i. to get an ODE in $\pi_1(U_1^+)$.
- Use this and 4.i. to analyze the existence, uniqueness and regularity of solutions of the ODE in $\mathbb{S}^2$ for $g(x) = \langle x, Ax\rangle$, for $x \in \mathbb{S}^2$, where $A \in M_{2\times 2}(\mathbb{R})$.

ii. **Consider** the function $g(x) = \frac{1}{d(x,p_1)} + \frac{1}{d(x,p_2)}$ for $x \in \mathbb{S}^2 \setminus \{p_1, p_2\}$, where $p_1 \neq p_2 \in \mathbb{S}^2$ and d is the spherical distance (length of the great circle joining the two points). For which initial conditions do solutions exist? Are they unique? Describe the solutions.

(b) <u>Flow of vector fields in the unit ball with compactly supported radial component</u> Let $X : B(0,1) \to \mathbb{R}^n$ $X(x) = (X_r(x), X_\theta(x))$ be a vector field expressed in polar-coordinates

$$X_r \in C^1_c(B(0,1), \mathbb{R}), \quad X_\theta \in C^1(B(0,1), \mathbb{R}^{n-1}),$$

so that for the radial unit vector e_r at x,

$$\langle (X_r, 0, \ldots, 0), e_r\rangle = X_r(x), \quad \text{and} \quad \langle (0, X_\theta(x)), e_r\rangle = 0.$$

Show that for any initial condition in $B(0,1)$, solution exists for all times, is unique, and the flow is C^1.

(c) <u>Flows and 1-parameter families of diffeomorphisms in compact manifolds</u>
Show that if X is a C^1-vector field on a smooth compact manifold, then we get a one-parameter family of diffeomorphisms.

(d) <u>Linear time-dependent increments</u>
Let $V \subset \mathbb{R}^n$ be open. **Show** that if

$$A : [-\epsilon_0, \epsilon] \times V \to M_{n\times n}(\mathbb{R})$$

is a C^1 map, then for any $x_0 \in V$, there are

$$\epsilon = \epsilon(x_0, A) > 0, \quad \delta = \delta(x_0, A) > 0,$$

such that there is a C^1 map $\Phi : [-\epsilon, \epsilon] \times B(x_0, \delta) \to M_{n\times n}(\mathbb{R})$, such that for each $x \in B(x_0, \delta)$, the map $t \to \Phi(t, x)$ solves

$$\frac{d}{ds}\Big|_{s=t} \Phi(s, x) = A(t, x)\Phi, \quad \Phi(0, x) = id.$$

<u>Remark</u>: Note in this problem that the initial conditions are from an Euclidean subspace whereas the image is not in the same subspace, but in some linear space. To solve this problem use the techniques of Picard iteration on the integral form of the ODE and Gronwall's inequality. This problem is taken from Perko, and the result was used in the proof of C^1 regularity of solutions wrt initial conditions.

(e) Failure of uniqueness
Show that the ODE

$$\frac{d}{ds}\Big|_{s=t}\gamma(s) = (\gamma(t))^{2/3}, \quad \gamma(0) = 0,$$

does not have a unique solution. Note that the Lipschitz property fails at $t = 0$, which already says that uniqueness might fail, and here it does.

(f) Blow-up at a finite-time
For the ODE

$$\frac{d}{ds}\Big|_{s=t}\gamma(s) = (\gamma(t))^{2}, \quad \gamma(0) = 1$$

show that a unique solution exists and determine the domain of existence of the solution. Show that $\gamma(t) \to \infty$ as $t \to 1^-$.

(g) Gronwall gives sharp bounds Construct an ODE where the deviation upper bound obtained from Gronwall is achieved, that is the deviation grows at least linearly in distance between initial conditions and exponentially in time.