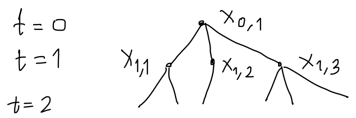
A Proof of Cayley's Formula via Poisson Branching Process

Friday, 29 September, 2023 11:28 PM

Bikram Halder

Branching Process

- One of the simplest model for a populat^n evolving with time.
- Consider, particles such as bacteria that can generate particles of same type.



$X_{t,i}$ = # of children of i^{th} particle at time t

$X_{t,i} \overset{iid}{\sim} X$ ($\mathbb{N} \cup \{0\}$ - valued r.v.)

Total # of offspring generated at time t, $Z_t = \sum_{i=1}^{Z_{t-1}} X_{t,i}$

Q. What is the chance that the populat^n dies out? or survive forever?

Denote, $\eta = \mathbb{P}(\exists t \text{ s.t. } Z_t = 0)$

Recall. (pgf) $G_X(s) = \mathbb{E}[s^X]$

$$= \sum_{r \ge 0}^{\infty} s^r \, \mathbb{P}(X = r)$$

$G_X^{(r)}(0) = r! \, \mathbb{P}(X = r)$

$\Rightarrow$ Well-defined & unique

Thm (Survival vs Extinction for BP)

For a B.P. with offspring dist^n iid X,

(i) $\eta = 1$ if $\mathbb{E}[X] < 1$

(ii) $\eta < 1$ if $\mathbb{E}[X] > 1$

(iii) $\eta = 1$ if $\mathbb{E}[X] = 1$ & $\mathbb{P}(X = 1) < 1$

In other words,

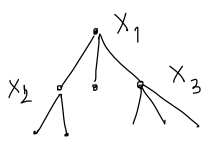
$\eta = \inf \{ s \in [0,1] \mid G_X(s) = s \}$

pgf

Random Walk Perspective & Law of Total Progeny

$T = \sum_{n=0}^{\infty} Z_n$

We do the same thing but instead of splitting generation-wise we do it leaf-wise.



$(X_i)_{i=1}^{n} \overset{iid}{\sim} X$

Same q^n of extinction!

At each time t, we want to count the $\underbrace{\text{\# of active particles}}_{S_t}$

X_t = # of particles generated at time t (iid)

$S_0 = 1$, $\quad S_t = X_t + (S_{t-1} - 1)$ the parent who died

$\quad = X_1 + \cdots + X_t - (t-1)$

$T = \inf\{t \mid S_t = 0\}$

$\quad = \inf\{t \mid X_1 + \cdots + X_t = t-1\}$

(Can be interpreted as total progeny of BP)

Thm $\mathbb{P}(T=n) = \frac{1}{n}\mathbb{P}(X_1 + \cdots + X_n = n-1)$

proof. Follows from Kemperman's Formula (link)

Poisson Dist^n $\quad X \sim Poi(\lambda)$

Recall. $\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$X_1, \ldots, X_n \overset{iid}{\sim} Poi(\lambda)$

$X_1 + \cdots + X_n \sim Poi(n\lambda)$

Then, using above thm,

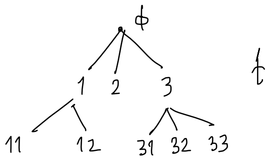
$$\mathbb{P}(T=n) = \frac{1}{n}\mathbb{P}(X_1 + \cdots + X_n = n-1)$$

$$= \frac{1}{n} e^{-n\lambda}\frac{(n\lambda)^{n-1}}{(n-1)!} = \frac{e^{-n\lambda}\lambda^{n-1}}{(n-1)!} \cdot n^{n-2}$$

Proof of Cayley's Formula

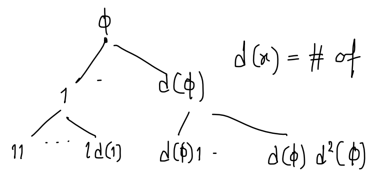
Introduce,

Family trees



Ulam-Harris' Representation $\longrightarrow$

For $w \in t$, $|w|$ = distance from ϕ



$d(x)$ = # of children of x

Observe,

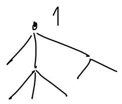
$(d(w))_w$ = Collect^n of iid r.v $\Rightarrow$ BP

$\mathcal{T}$ = family tree of a BP. with $X \sim Poi(1)$ offspring dist^n.

$$\mathbb{P}(\mathcal{T} = t) = \prod_{w \in t} \mathbb{P}(X = d(w)) = \prod_{w \in t} \frac{e^{-1} 1^{d(w)}}{d(w)!} = \frac{e^{-|t|}}{\prod_{w \in t} d(w)!}.$$

Conditionally, on total progeny $T^* = n$ & family tree $\mathcal{T}$,

We introduce a labelling on $\mathcal{T}$ as,



Fix 1 & give all other vertices

labelling from $\{2, .. , n\}$ u.a.r. without replacement.

Fix a label l on family trees t, [$t \sim l$ if t can be given label l]

$$\# \{t \mid t \sim l\} = \frac{\prod_{w \in t} d(w)!}{L(l)} \qquad L(l) = \# \{ l' \mid t \sim l' \,\&\, l' \simeq l \}$$

(doesn't depend on t; same $\forall t$)

= # family trees compatible with l

$$= \frac{\prod_{w \in l} d_w!}{L(l)} \qquad d_w = \begin{cases} \deg(w) - 1 , & w \neq 1 \\ \deg(w) , & w = 1 \end{cases}$$

$$\mathbb{P}(t \text{ recieves label } l) = \frac{L(l)}{(|l| - 1)!}$$

Hence,

When labelling a family tree of a Poisson Branching Process with parameter 1,

Prob of obtaining a given labelled tree l

$$\mathbb{P}(\mathcal{L} = l) = \sum_{t \sim l} \mathbb{P}(\mathcal{T} = t, \ t \text{ recieves label } l)$$

$$= \sum_{t \sim l} \mathbb{P}(\mathcal{T} = t \mid t \text{ recieves label } l) \, \mathbb{P}(t \text{ recieves label } l)$$

$$= \sum_{t \sim l} \mathbb{P}(\mathcal{T} = t) \, \mathbb{P}(t \text{ recieves label } l)$$

$$= \sum_{t \sim l} \frac{e^{-n}}{\prod_{w \in l} d(w)!} \cdot \frac{L(l)}{(|l| - 1)!}$$

$$= \# \{t \mid t \sim l\} \frac{e^{-|l|}}{\prod_{w \in l} d(w)!} \cdot \frac{L(l)}{(|l| - 1)!}$$

$$= \#\{t \mid t \sim \ell\} \frac{e^{-|\ell|}}{\prod_{w \in L} d(w)!} \cdot \frac{L(\ell)}{(|\ell|-1)!}$$

$$= \frac{\prod_{w \in \ell} d_w!}{L(\ell)} \cdot \frac{e^{-|\ell|}}{\prod_{w \in L} d(w)!} \cdot \frac{L(\ell)}{(|\ell|-1)!} = \frac{e^{-|\ell|}}{(|\ell|-1)!}$$

$$\mathbb{P}(\mathcal{L} = \ell \mid |\mathcal{L}| = n) = \frac{\mathbb{P}(\mathcal{L} = \ell, |\mathcal{L}| = n)}{\mathbb{P}(|\mathcal{L}| = n)} = \frac{\mathbb{P}(\mathcal{L} = \ell)}{\mathbb{P}(|\mathcal{L}| = n)}$$

$$= \frac{e^{-n}}{(n-1)!} \cdot \frac{(n-1)!}{e^{-n} n^{n-2}} = \frac{1}{n^{n-2}}$$

$$\Rightarrow \boxed{\mathcal{L} \mid |\mathcal{L}| = n \sim \mathrm{Unif}[n^{n-2}]}$$

References:

1. Random Graphs and Complex Networks, Volume I ~ Remco van der Hofstad
   https://www.win.tue.nl/~rhofstad/NotesRGCN.pdf
2. The Random Walk Construction of Uniform Spanning Trees and Uniform Labelled Trees ~ David J Aldous
   https://www.cs.cmu.edu/~15859n/RelatedWork/AldousRandomTrees.pdf