Thresholds and Expectation thresholds

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1. Random Graph
2. Thresholds
3. Kahn-Kalai Conjecture – Proved by Pham & Park

1. Random Graph

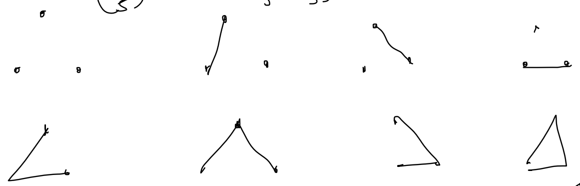
$G_{n,p}$ (Erdős-Rényi random-graph)

vertex set $[n] = \{1, \dots, n\}$ where n is large by fixed

each edge exist with probability p

$G_{3,1/2}$

$\left(\frac{4}{5}\right)^3$ $\quad$ $\frac{1}{5}\left(\frac{4}{5}\right)^2$



$\frac{1}{5^2}\left(\frac{4}{5}\right)$ $\quad$ $\left(\frac{1}{5}\right)^3$

each of these graphs exist w.p. $\frac{1}{2^3} = \frac{1}{8}$

$G_{3,1/5}$

Q: $P(G_{n,p}$ is conneed$)$?

$\mathbb{P}(G_{n,p}$ is 3-colourable$)$?

$\mathbb{P}(G_{n,p}$ contain fixed graph$)$?

Typically, Largest conn. component size $\begin{cases} \lesssim \log n & p < \frac{1-\varepsilon}{n} \\ \simeq n & p > \frac{1+\varepsilon}{n} \end{cases}$

$L \asymp n$ $\quad p > \frac{1+\epsilon}{n}$

at $p = \frac{1}{n}$ $\quad \asymp n^{3/2}$

Qn Threshold for various properties

Small subgraphs H (Erdős - Renyi '59, Bollobás 81) $p^* = n^{-1/m(H)}$ — maximum subgraph den $= \max\{\text{densn}(\tilde{H}) : \tilde{H} \subseteq H\}$

Connectivity (") $\frac{\log n}{n}$

Perfect matching (" '66) $\frac{\log n}{n}$

Ramsey property

Turan "

Colourability

⋮

Take X a finite set $\quad 2^X$ — power set

$$\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|} \quad \text{for } A \subseteq X$$

$X_p \sim \mu_p$ "p-random subset of X"

$X_p = E(K_n) = \binom{[n]}{2}$, $\quad X_p \overset{d}{=} G_{n,p}$

$X_p = \mathcal{A}$ follow property that an edge exist

Say, $X = \{1,2,3\}$

$\{1,2,3\}$

$\{1,2\}$ $\{2,3\}$ $\{1,3\}$

$\{1\}$ $\{2\}$ $\{3\}$

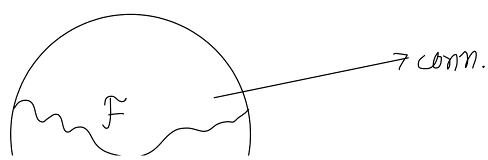
ϕ

Property, $\mathcal{P}$ is subcollection of the 2^X

$n = 3$

An increasing property, $\mathcal{F}$ is a property such that if $A \in \mathcal{F}$ & $A \subseteq B$ then $B \subseteq \mathcal{F}$

X



Thm. For any increasing property $\mathcal{F}$ (non-trivial ϕ, 2^X)

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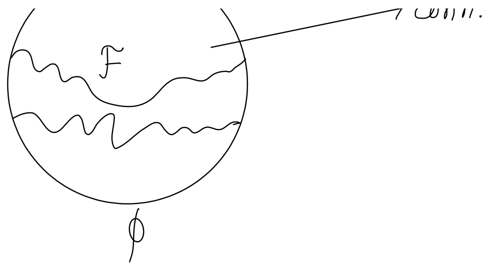
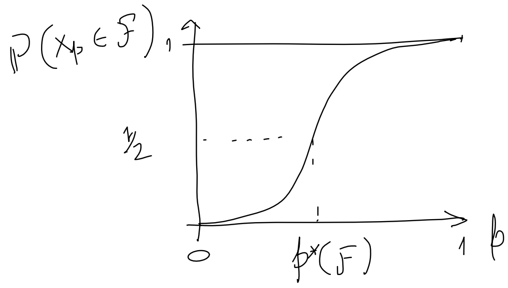
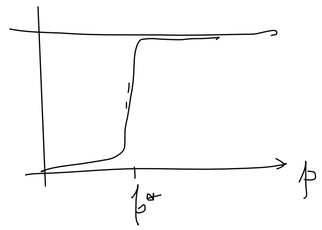
$\mu_p(\mathcal{F}) = \mathbb{P}(X_p \in \mathcal{F})$ is cts & strictly increasing



eg. $\mathcal{F}$ = graph conn. $\left(X = \binom{[n]}{2} \right)$

Threshold^n





↪ threshold for $\mathcal{F}$

↓

Locat^n of the threshold

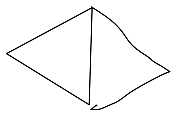
- Kahn-Kalai Conjecture ('06) - a very strong result
  - suggest a general bound

Sharpness -

Kahn-Kalai-Linial ('88), Friedgut-Kalai ('96), Friedgut ('99)
(based on Fourier analysis)

Locat^n of Threshold

Eg Threshold for containment of



H_1 $\quad m(H_1) = d(H_1) = \frac{5}{4}$

$X = \binom{[n]}{2}$ $\quad (X_p = G_{n,p})$ $\quad$ ↳ property - $\mathcal{F}_H$

$$\mathbb{E}[\#\{H \subseteq G_{n,p}\}] \asymp n^4 p^5 \longrightarrow \begin{cases} 0 & p \ll n^{-4/5} \\ \infty & p \gg n^{-4/5} \end{cases}$$

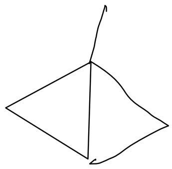
threshold for $\mathcal{F}$ $\asymp$ $n^{-4/5}$

$$\mathbb{P}(\#\{H \subseteq G_{n,p}\} > 0) \le \mathbb{E}(\cdots)$$

$p^*(\mathcal{F}_{H_1}) \gtrsim n^{-4/3}$

$p^*(\mathcal{F}_{H_1}) \asymp n^{-4/5}$

Q? Does $\mathbb{E}$-threshold predict $p^*(\mathcal{F}_H)$?

Eg. Threshold for containment of  $- H_2$

$m(H_2) = \frac{5}{4}$

$= \{\text{density}(\tilde{H}) \mid \tilde{H} \subseteq H_2\}$

$$\mathbb{E}[\#\{H_2 \subseteq G_{n,p}\}] \asymp n^5 p^6 \longrightarrow \begin{cases} 0 & p \ll n^{-5/6} \\ \infty & p \gg n^{-5/6} \end{cases}$$

"$\mathbb{E}$-threshold" $\asymp n^{-5/6}$

$p^*(\mathcal{F}_{H_2}) \gtrsim n^{-5/6}$

$p^*(\mathcal{F}_{H_2}) \asymp n^{-4/5}$

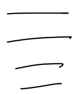
Thm (Erdős-Renyi, Bollobás)

For any fixed graph H,

$p^*(\mathcal{F}_H) \asymp$ "$\mathbb{E}$-threshold" of the "densest" subgraph of H

$\asymp n^{-1/m(H)}$

max subgraph density

Eg. Existance of perfect matching 

n even

$X = \binom{[n]}{2}$, $X_p = G_{n,p}$ $\quad \mathcal{F}$ = contain a perfect matching

"$\mathbb{E}$-threshold" $\asymp \frac{1}{n}$

$p^*(\mathcal{F}) \gtrsim \frac{1}{n}$

$p^*(\mathcal{F}) \asymp \frac{\log n}{n}$

Total n-coupons

Each box contain 2 coupons

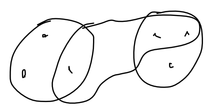
What is expected # of boxes you need to pick to collect all the coupon $\lesssim n \log n$

$$\binom{n}{2} p \lesssim n \log n \Leftrightarrow p \lesssim \frac{\log n}{n}$$

Eg. Perfect matching in Hypergraph (Shamir's problem '80)

" (V, E) where $E \subseteq 2^V$

$X = \binom{[n]}{r}$



X_p = random r-uniform hypergraph $\mathcal{H}^r_{n,p}$

3-unif hypergraph

$r = 2$, Erdős-Rényi ('66)

$r \geq 3$

$$\mathbb{E}[\#\text{ p.m. in } \mathcal{H}^r_{n,p}] \asymp (n^2 p)^{n/3}$$

$\Rightarrow$ "$\mathbb{E}$-threshold" $\asymp 1/n^2$

$\Rightarrow$ $p^*(\mathcal{F}) \gtrsim \frac{1}{n^2}$

& because of the coupon collector behavior $p^*(\mathcal{F}) \gtrsim \frac{n \log n}{n^3}$

$\asymp \frac{\log n}{n^2}$

$p^*(\mathcal{F}) \asymp \frac{\log n}{n^2}$ (Johansson-Kahn-Vu '08)

In general, $p^*(\mathcal{F}) \gtrsim \frac{\log n}{n^{r-1}}$

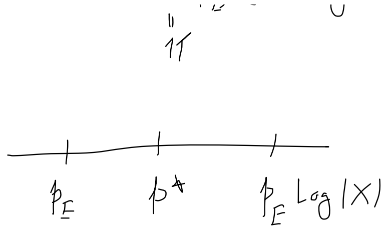
& "$\mathbb{E}$-threshold" $\simeq n^{-(r-1)}$

Kahn-Kalai-Conjecture (Proved by Pham-Park '22) universal

For any increasing property $\mathcal{F}$, $p^*(\mathcal{F}) \leq \underset{\substack{\| \\ 17}}{K}\, p_E(\mathcal{F}) \log |X|$

$(b / c) \leq)$

$\left(p_E(\mathcal{F}) \leq \right)$



Apply KKC to $p^*(\mathcal{F}) \lesssim \frac{\lg n}{n^{r-1}}$

then we get $p^*(\mathcal{F}) \asymp \frac{\lg n}{n^{r-1}}$

Threshold for bounded-degree spanning trees (Montgomery '19)

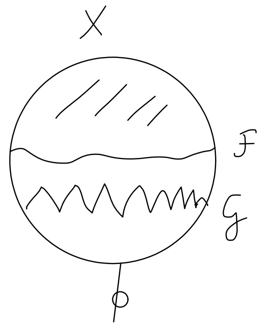
$\mathcal{F} \subseteq 2^X$ $\quad p_E(\mathcal{F}) = \max\{q : \exists \mathcal{G}\} \leq p^*(\mathcal{F})$

$p^*(\mathcal{F}) \geq q$ if $\exists\, \mathcal{G} \subseteq 2^X$ s.t.

(i) "$\mathcal{G}$ covers $\mathcal{F}$": $\forall A \in \mathcal{F}\ \exists$ s.t. $A \supseteq B$

(ii) $\sum_{S \in \mathcal{G}} q^{|S|} \leq \frac{1}{2}$

"$= \mathbb{P}(X_q \in \mathcal{G})$"



eg $\mathcal{F}$: containment of

$\mathcal{G}_1 = \{$ All (labelled) copies of $\}$

$\longrightarrow \sum_{S \in \mathcal{G}_1} q^{|S|} \leq \frac{1}{2}$ for $q \lesssim n^{-4/5} \Rightarrow p^*(\mathcal{F}) \gtrsim n^{-4/5}$

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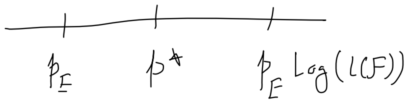
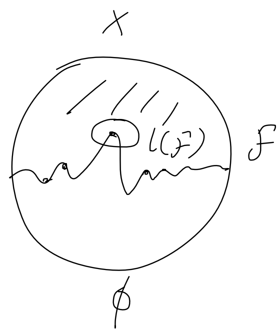
Fractional version (Talagrand '10, proved by Frankston-Kahn-Narayanan-Park '19)

universal

For any increasing property $\mathcal{F}$, $p^*(\mathcal{F}) \leq K\, p_E(\mathcal{F}) \log(\ell(\mathcal{F}))$

$\left(p_E(\mathcal{F}) \leq \right)$





Largest minimal element

References:

1. A Proof of the Kahn-Kalai Conjecture ~ Jinyoung Park, Huy Tuan Pham
   https://arxiv.org/abs/2203.17207

2. Thresholds and expectation thresholds ~ Jeff Kahn, Gil Kalai
   https://arxiv.org/abs/math/0603218

3. Threshold Phenomena for Random Discrete Structure ~ Jinyoung Park
   https://www.ams.org/notices/202310/rnoti-p1615.pdf